Math 103 Day 13: The Mean Value Theorem and How Derivatives Shape a Graph

Ryan Blair

University of Pennsylvania

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Outline

Theorem

(Rolle's Theorem) Let f be a function that satisfies the following three hypothesis:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$¿

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Example: $f(x) = 1 - x^2$ on $[-1, 1]$.

Example Verify that the function $f(x) = 5 - 12x + 3x^2$ satisfies the hypothesis of Rolle's Theorem on $[1, 3]$. Then find all c in $[1, 3]$ such that $f'(c) = 0$.

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$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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ExampleVerify that $f(x) = 3x^2 + 2x + 5$ satisfies the hypothesis of the Mean Value Theorem on $[-1, 1]$. Then find all numbers c satisfying the conclusion of the Mean Value Theorem.

Exercise

Show that $f(x) = x^3 - 15 + c$ has at most one real root in $[-2, 2]$.

How Derivatives Shape a Graph

Increasing/Decreasing Test

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

First Derivative Test

Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f does not change sign at c, then f has no local maximum or minimum at c.

Definition

If a graph of f lies above all of its tangents on an interval I, then is is called **concave up** on I. If a graph of f lies below all of its tangents on an interval I, then is is called **concave down** on I.

Concavity test

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Definition

A point P on a curve $y = f(x)$ is called and **inflection point** if f is continuous there and the curve changes from concave down to concave up or from concave up to concave down at P.

The Second Derivative Test

Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.